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THE PLUS CONSTRUCTION, POSTNIKOV TOWERS AND UNIVERSAL CENTRAL MODULE EXTENSIONS

BY

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ABSTRACT

Given a connected space X, we consider the effect of Quillen's plus construction on the homotopy groups of X in terms of its Postnikov decomposition. Specifically, using universal properties of the fibration sequence $AX \to X \to X^+$, we explain the contribution of $\pi_n X$ to $\pi_n X^+$, $\pi_{n+1} X^+$ and $\pi_n AX$, $\pi_{n+1} AX$ explicitly in terms of the low dimensional homology of $\pi_n X$ regarded as a module over $\pi_1 X$. Key ingredients developed here for this purpose are universal Π-central fibrations and a theory of universal central extensions of modules, analogous to universal central extensions of perfect groups.

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Introduction

Quillen's plus construction (cf. [10]), applied to a space X, yields a universal map $\eta\colon X \to X^+$, which is characterized by the fact that it quotients out the maximal perfect subgroup of $\pi_1 X$ and induces isomorphisms in all homology theories (including homology with twisted coefficients). In general, a map between connected spaces satisfies this homological condition if and only if its homotopy fiber is acyclic; see [11] and compare [5]. We denote the homotopy fiber of $\eta\colon X \to X^+$ by AX.

Understanding the map $\pi_*\eta\colon \pi_*X \to \pi_*X^+$ is helpful in studying the effect of homological localization functors on homotopy groups (see 4.5) and in higher algebraic K-theory. Such understanding was obtained early on in low dimensions and, except for special cases, this has remained the extent of our knowledge. With the following result we clarify completely the contribution of $\pi_n X$ to $\pi_n X^+$ and $\pi_{n+1}X^+$, for each $n \geq 2$.

THEOREM A: *Let X be a connected CW complex. Applying the plus construction to the Postnikov section $K(\pi_n X, n) \to P_n X \to P_{n-1}X$ ($n \geq 2$) yields the commutative diagram of fibrations whose properties are formulated below:*

(FD)
$$\begin{array}{ccccc} F & \longrightarrow & K(\pi_n X, n) & \longrightarrow & \Phi \\ \downarrow & & \downarrow & & \downarrow \\ AP_nX & \longrightarrow & P_nX & \longrightarrow & (P_nX)^+ \\ \downarrow & & \downarrow & & \downarrow \\ AP_{n-1}X & \longrightarrow & P_{n-1}X & \longrightarrow & (P_{n-1}X)^+ \end{array}$$

The fibers F and Φ are $(n-1)$-connected, and their lowest non-vanishing homotopy groups fit into the natural commutative diagram of exact sequences in which every vertical arrow is an isomorphism.

(UCE)
$$\begin{array}{ccccccc} H_1(\widetilde{G};\pi_nX) & \rightarrowtail & I[\widetilde{G}]\otimes_{\widetilde{G}}\pi_nX & \xrightarrow{\mu} & \pi_nX & \twoheadrightarrow & H_0(\widetilde{G};\pi_nX) \\ \cong\downarrow & & \cong\downarrow & & \| & & \downarrow\cong \\ \pi_{n+1}\Phi & \rightarrowtail & \pi_nF & \longrightarrow & \pi_nX & \twoheadrightarrow & \pi_n\Phi \end{array}$$

Moreover, there is an epimorphism

$$\pi_{n+2}\Phi \xrightarrow{\cong} \pi_{n+1}F \twoheadrightarrow H_{n+1}F \xleftarrow{\cong} H_2(\widetilde{G}; I[\widetilde{G}]\otimes_{\widetilde{G}}\pi_nX).$$

Here $\widetilde{G}$ is the universal central extension of the maximal perfect subgroup G of $\pi_1 X$, and $I[\widetilde{G}]$ is the augmentation ideal of the integral group ring of $\widetilde{G}$.

ON THE BACKGROUND OF THEOREM A. Our approach to Theorem A is guided by properties of the homotopy fibration sequence of η:

$$\Omega X^+ \xrightarrow{i} AX \longrightarrow X \xrightarrow{\eta} X^+.$$

As noted in [9, 0.1.iv], the acyclic space AX is just the acyclization of X, as defined by Dror in [3] (see also [4]). The following two theorems express the universal properties of the plus construction and acyclization in a form which lends itself better to an interpretation in terms of homotopy groups.

THEOREM B ([9, 7.7]): *The fibration $\Omega X^+ \xrightarrow{i} AX \to X$ is Π-central, in the sense that all Whitehead products $[i_*\alpha, \beta]$ vanish, where $\alpha \in \pi_p \Omega X^+$ and $\beta \in \pi_q AX$, $p, q \geq 1$.*

THEOREM C: *The fibration $\Omega X^+ \to AX \to X$ is initial amongst Π-central fibrations in the following sense: given a solid diagram of Π-central fibrations*

$$\begin{array}{ccccc} \Omega X^+ & \longrightarrow & AX & \longrightarrow & X \\ \vdots & & \vdots & & \| \\ \downarrow & & \downarrow & & \| \\ F & \longrightarrow & E & \xrightarrow[q]{} & X \end{array}$$

in which $G := \operatorname{im}(q)$ is the maximal perfect subgroup of $\pi_1 X$, dotted maps exist making the diagram commute. Moreover, the dotted maps are unique up to vertical homotopy.

To get a feel for the implications of Theorems B and C, consider first the following well known exact sequence

$$\begin{array}{ccccccc} \pi_2 X^+ & \longrightarrow & \pi_1 AX & \longrightarrow & \pi_1 X & \longrightarrow & \pi_1 X^+ \\ & \searrow \quad \nearrow & & \searrow \quad \nearrow & & & \\ & H_2 G & & G & & & \end{array}$$

in which $\pi_1 AX$ is the universal central extension of G.

This sequence can be nicely explained as a consequence of Theorems B and C, using results on the universal central extension of a perfect group, due to Milnor [8, Sect. 5] and Kervaire [6].

As another consequence of Theorems B and C, we obtain Theorem A. It depends upon a new concept from algebra, namely the universal central extension of a perfect module:

THEOREM D: *If G is a 2-acyclic group (that is, $H_1(G;\mathbb{Z}) = 0 = H_2(G;\mathbb{Z})$), then every G-module M fits into an exact sequence*

$$H_1(G;M) \rightarrowtail I[G] \otimes_G M \xrightarrow{\mu} M \twoheadrightarrow H_0(G;M)$$

whose terms have the following properties:

(i) $\mathrm{im}(\mu) = I[G].M$ *is the unique maximal perfect submodule of M; i.e.,* $I[G].\mathrm{im}(\mu) = \mathrm{im}(\mu)$.

(ii) $H_1(G;M) \rightarrowtail I[G] \otimes_G M \twoheadrightarrow \mathrm{im}(\mu)$ *is a central extension of* $\mathrm{im}(\mu)$ *(i.e., G acts trivially on $H_1(G;M)$), and it is initial amongst all such central extensions.*

ORGANIZATION OF THE PAPER. Section 1 supplies some facts on Π-central fibrations, leading up to Theorem C. Section 2 develops material on universal central extensions of a module over a group ring, leading up to Theorem D. In Section 3 we prove Theorem A, and in Section 4 we compute the acyclic Postnikov invariants of AX (cf. [3]) in terms of the ordinary Postnikov invariants of X.

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1. Π-central fibrations with perfect target

Here we develop properties of Π-central fibrations, leading up to Theorem C which, in turn, guides our approach towards analyzing the effect of the plus construction on homotopy groups. We assume throughout that spaces, maps and homotopies are pointed. Spaces are assumed to be path connected, except possibly those arising as homotopy fibers.

1.1 Definition [9, Sect. 7]: A fibration sequence $F \xrightarrow{i} E \to B$ is called **Π-central** if all Whitehead products $[i_*\alpha, \beta]$ vanish for any $\alpha \in \pi_p F$ and $\beta \in \pi_q E$ with $p, q \geq 1$.

Given a map $q\colon W \to Y$, we refer to $G := \mathrm{im}(q) < \pi_1 Y$ as its **target** in homotopy dimension 1. We say that q has **perfect target** (in homotopy dimension 1) if G is a perfect group.

1.2 LEMMA: *Let $F \to W \xrightarrow{q} Y$ be a Π-central fibration, such that q has perfect target $G < \pi_1 Y$, and suppose $f\colon X \to Y$ is a map for which $G' := (\pi_1 f)^{-1} G$ is a* *perfect subgroup of* $\pi_1 X$. *Then the pullback fibration*

$$\begin{array}{ccccc} F & \xrightarrow{i'} & W' & \xrightarrow{q'} & X \\ \| & & \downarrow{\scriptstyle f'} & \text{pull back} & \downarrow{\scriptstyle f} \\ F & \xrightarrow[i]{} & W & \xrightarrow[q]{} & Y \end{array}$$

is Π*-central, and* q' *has perfect target* G'.

Proof: The pullback of a Π-central fibration is again Π-central by [9, 7.6]. An elementary argument shows that $\mathrm{im}(\pi_1 q') = G'$, which is perfect by assumption. ∎

1.3 Example: For every connected CW-space X, the sequence $\Omega X^+ \to AX \xrightarrow{c} X$ is a Π-central fibration, and c has perfect target equal to the maximal perfect subgroup of $\pi_1 X$.

1.4 Definition: A Π-central fibration $F \to W \xrightarrow{q} X$ such that q has perfect target G is **universal** if, under conditions (i) and (ii) below, for every solid diagram

$$\begin{array}{ccccc} F & \longrightarrow & W & \xrightarrow{q} & X \\ \vdots & & \vdots{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ F_1 & \longrightarrow & W_1 & \xrightarrow[q_1]{} & Y \end{array}$$

there exists a morphism $\tilde{f}$ of fibrations, unique up to homotopy, which makes the diagram commute. Diagram conditions:

(i) The bottom row is a Π-central fibration such that q_1 has perfect target G_1;
(ii) $f_*(G)$, the image of G under $\pi_1 f$, is contained in G_1.

1.5 Theorem: *For every connected CW-space* X, *the* Π*-central fibration* $\Omega X^+ \to AX \xrightarrow{c} X$ *is universal, and* c *has target* G *(=the maximal perfect subgroup of* $\pi_1 X$*).*

Proof: According to Definition 1.4, suppose we are given a solid diagram

$$\begin{array}{ccccc} \Omega X^+ & \longrightarrow & AX & \xrightarrow{c} & X \\ \vdots & & \vdots{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ F_1 & \longrightarrow & W_1 & \xrightarrow[q_1]{} & Y \end{array}$$

We use obstruction theory to obtain the required morphism of fibrations. At the level of fundamental groups we have the diagram of central extensions

$$\begin{array}{ccccc} H_2G & \rightarrowtail & \pi_1 AX & \xrightarrow{\pi_1 c} & G \\ \vdots & & \vdots & & \downarrow{\scriptstyle \pi_1 f} \\ \ker(\pi_1 q_1) & \rightarrowtail & \pi_1 W_1 & \twoheadrightarrow & G_1 \end{array}$$

with universal top row. Thus there exists a lift $\widetilde{f}^2\colon (AX)^2 \to W_1$ from the 2-skeleton of AX to W_1, and its restriction to $(AX)^1$ is homotopically unique. The existence and homotopical uniqueness of $\widetilde{f}$ follow because AX is acyclic and the action of $\pi_1 Y$ on $\pi_* F_1$ is trivial. ∎

Theorem C follows as a special case of Theorem 1.5.

2. Universal central extensions of perfect G-modules

In this section we develop the concept of perfect modules and their central extensions, and prove Theorem D. We assume some background material on perfect groups and their universal central extensions from [8, Sect. 5].

Given a group G and a left G-module M, we often use the exact "multiplication sequence"

$$\text{(MS)} \qquad H_1(G;M) \rightarrowtail I \otimes_G M \xrightarrow{\mu} M \twoheadrightarrow H_0(G;M)$$

which comes from applying $\mathrm{Tor}_{-}^{\mathbb{Z}[G]}(-,M)$ to $I \rightarrowtail \mathbb{Z}[G] \twoheadrightarrow \mathbb{Z}$. Here $\mathbb{Z}[G]$ is the integral group ring of G and I is its augmentation ideal. All tensor products are over $\mathbb{Z}[G]$, and μ is the multiplication map.

2.1 Definition: For $n \geq 1$, a group G is called n-**acyclic** if $H_k(G;\mathbb{Z}) = 0$ for $1 \leq k \leq n$.

Thus 1-acyclic groups are known as perfect groups. 2-acyclic groups are sometimes called "superperfect".

2.2 Definition: Let G be a group and $n \geq 0$. A G-module M is called n-**acyclic** if $H_k(G;M) = 0$ for $0 \leq k \leq n$.

In analogy with the group theoretic terminology, we sometimes refer to a 0-acyclic G-module as a "perfect G-module".

2.3 Lemma: *A group G is n-acyclic if and only if its augmentation ideal I is an $(n-1)$-acyclic G-module.*

Proof: Apply $H_*(G;-)$ to $I \rightarrowtail \mathbb{Z}[G] \twoheadrightarrow \mathbb{Z}$. ∎

2.4 Corollary: *A group G is 1-acyclic if and only if the multiplication map $\mu\colon I \otimes_G I \to I$ is an epimorphism. G is 2-acyclic if and only if μ is an isomorphism.*

Proof: Apply Lemma 2.3 to (MS), using $M = I$. ∎

2.5 Corollary: *For $n = 1, 2$ let G be an n-acyclic group, and let M be an arbitrary G-module; then the G-module $I \otimes_G M$ is $(n-1)$-acyclic.*

Proof: The multiplication map μ' for $M' = I \otimes_G M$ is given by the composite

$$I \otimes_G (I \otimes_G M) \xrightarrow{\cong} (I \otimes_G I) \otimes_G M \xrightarrow{\mu_I \otimes M} I \otimes_G M.$$

Thus the claim follows from Corollary 2.4. ∎

2.6 Corollary: *If G is an n-acyclic group and A is an abelian group with trivial G-action, then $\mathrm{Tor}_k^{\mathbb{Z}[G]}(I, A) = 0$, for $0 \le k \le n-1$.*

Proof: Use the long exact sequence obtained by applying $\mathrm{Tor}_*^{\mathbb{Z}[G]}(-, A)$ to $I \rightarrowtail \mathbb{Z}[G] \twoheadrightarrow \mathbb{Z}$. ∎

2.7 Definition: A **colocalizing functor** on a category $\mathcal{C}$ is a functor $C\colon \mathcal{C} \to \mathcal{C}$, together with a natural transformation $\varepsilon\colon C \to \mathrm{Id}_{\mathcal{C}}$ making the diagram below commutative.

$$\begin{array}{ccc} C \circ C & \xrightarrow[\cong]{C\varepsilon} & C \\ {\scriptstyle \varepsilon \circ C}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \varepsilon} \\ C & \xrightarrow[\varepsilon]{} & \mathrm{Id} \end{array}$$

2.8 Theorem: *For a 2-acyclic group G, the functor $E := I \otimes_G -$, together with the natural transformation $\mu\colon E \to \mathrm{Id}$ defined by*

$$\mu_M\colon I \otimes_G M \xrightarrow{\text{multiply}} M,$$

is a colocalizing functor from the category $\mathbb{Z}[G]$-$\mathcal{M}$od of left G-modules onto the category $A_1\mathbb{Z}[G]$-$\mathcal{M}$od of 1-acyclic G-modules.

Proof: E takes values in $A_1\mathbb{Z}[G]$-$\mathcal{M}$od by Corollary 2.5. The colocalizing properties of E require that

(1) the diagram

$$\begin{array}{ccc} EEM & \xrightarrow[\cong]{E\mu_M} & EM \\ {\scriptstyle \mu_{EM}}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mu_M} \\ EM & \xrightarrow[\mu_M]{} & M \end{array}$$

be commutative and natural in M; and

(2) the designated arrows in this diagram be isomorphisms.

(1) follows from basic properties of the tensor product. For (2), use Corollary 2.4 to deduce that μ_{EM} is an isomorphism. To see that $E\mu_M = I \otimes_G \mu_M$ is an isomorphism, too, we break the sequence (MS) up into short exact sequences:

$$H_1(G;M) \rightarrowtail I \otimes_G M \overset{\mu_M}{\twoheadrightarrow} PM \quad \text{and} \quad PM \rightarrowtail M \twoheadrightarrow H_0(G;M).$$

Apply $\mathrm{Tor}_{-}^{\mathbb{Z}[G]}(I,-)$ to these sequences, and use Corollary 2.4 to see that $I \otimes_G \mu_M$ is the composite of the two isomorphisms

$$I \otimes_G (I \otimes_G M) \xrightarrow{\cong} I \otimes_G PM \quad \text{and} \quad I \otimes_G PM \xrightarrow{\cong} I \otimes_G M.$$

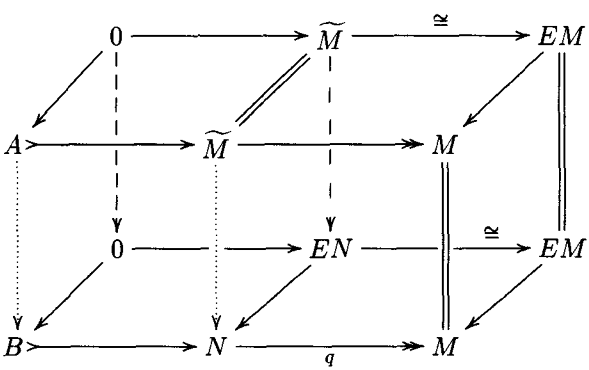
The claim follows. ∎

2.9 Definition: The **center** of a G-module N is the submodule of elements on which G acts trivially. A **central extension** of a G-module M is a short exact sequence of G-modules $A \rightarrowtail N \twoheadrightarrow M$ so that A maps into the center of N.

In analogy with universal central extensions of perfect groups we prove

2.10 THEOREM: *Given a 2-acyclic group G, a central extension $A \rightarrowtail \widetilde{M} \twoheadrightarrow M$ of G-modules is initial amongst all central extensions of M if and only if $\widetilde{M}$ is 1-acyclic.*

Proof: Assume $\widetilde{M}$ is 1-acyclic. In the diagram below, we assume the solid part of the front face is given.



The solid part of the back face results from applying the colocalizing functor E. We find $EA = 0 = EB$ by Corollary 2.4. Thus the back rows are exact, being the ends of $\mathrm{Tor}_*^{\mathbb{Z}[G]}(I, -)$-long exact sequences. So there is a map $\widetilde{M} \to EN$ which makes the right hand back square commute. This yields a map $f\colon \widetilde{M} \to N$ making the vertical square in the center, as well as the right front face, commute. To see that it is unique, assume $g\colon \widetilde{M} \to N$ is another such map. Then $q \circ (f - g)\colon \widetilde{M} \to M$ is the zero map, so $(f - g)$ lifts to B. This implies that $(f - g) = 0$, because $H_0(G; M) = 0$ and G acts trivially on B. Thus $f = g$, implying that the sequence is initial amongst all central extensions of M.

To see the converse, we invoke Theorem (2.11) which, of course, does not depend on the part of (2.10) we are going to prove now: Part (ii) shows that M is 0-acyclic. Part (i) implies that $\widetilde{M} \cong EM$ which is 1-acyclic. ∎

We call a sequence of G-modules, as in Theorem 2.10, the **universal central extension** of M.

2.11 Theorem: *Given a 2-acyclic group G, the following hold:*

(i) *For every 0-acyclic G-module M,*

$$H_1(G; M) \rightarrowtail I \otimes_G M = EM \twoheadrightarrow M$$

is a universal central extension of M.

(ii) *A G-module M has a universal central extension if and only if M is 0-acyclic; compare* [8, 5.7].

Proof: (i) The given sequence is (MS), taking into account that M is 0-acyclic. EM is 1-acyclic by Corollary 2.5. So the claim follows from Theorem 2.10.

(ii) Suppose M is not 0-acyclic, and $A \rightarrowtail N \twoheadrightarrow M$ is a universal central extension of M. Then M, and hence N, have $H_0(G; M) \neq 0$ as a G-trivial quotient. Therefore there are at least two distinct morphisms from the assumed universal central extension to the central extension

$$H_0(G; M) \rightarrowtail H_0(G; M) \oplus M \twoheadrightarrow M,$$

a contradiction. ∎

We remark that [7, Thm. 1] can be regarded as a precursor of Theorem 2.11.

Proof of Theorem D: (i) The module $I \otimes_G M$ is 1-acyclic by Corollary 2.5. So $PM = \mathrm{im}(\mu)$ is 0-acyclic by 2.12. It is a maximal 0-acyclic submodule of M because any module N with $PM < N < M$ yields a quotient $N/PM < H_0(G; M)$ with trivial G-action. However, N/PM is again perfect by Proposition 2.12

below. So $N = PM$. That PM is the unique maximal perfect submodule of M also follows from Proposition 2.12.

(ii) follows from Theorem 2.10. ∎

We conclude this section by formulating some closure properties of the classes of n-acyclic modules:

2.12 Proposition: *For any group G, and $n \geq 0$, the class of perfect G-modules is closed under quotients and arbitrary colimits.*

Proof: The natural isomorphism $H_*(G; \bigoplus_{\lambda\in\Lambda} M_\lambda) \cong \bigoplus_{\lambda\in\Lambda} H_*(G; M_\lambda)$ shows that the class of perfect G-modules is closed under direct sums. Further, any quotient M of a perfect G-module N is again perfect because $0 = H_0(G;N) \twoheadrightarrow H_0(G;M)$. ∎

2.13 Proposition: *Given a 2-acyclic group G, the class of 1-acyclic G-modules is closed under extensions and arbitrary colimits.*

Proof: If $M' \rightarrowtail M \twoheadrightarrow M''$ is an extension of G-modules with M' and M'' 1-acyclic, then inspection of the associated long exact sequence in homology shows that M is 1-acyclic as well. By Corollary 2.5, $I \otimes_G -$ takes values in the class of 1-acyclic G-modules. Moreover, $I \otimes_G -$ commutes with arbitrary colimits. ∎

3. Proof of Theorem A

By passing to the appropriate covering space of X, if necessary, we can assume that $\pi_1 X$ is perfect. So X^+ and each Postnikov section $(P_n X)^+$ ($n \geq 1$) are simply connected.

3.1 Lemma: *For $n \geq 2$, Φ is $(n-1)$-connected.*

Proof: This follows from the fact that, for $k \leq n$,

$$0 = H_k(P_{n-1}X, P_n X; \mathbb{Z}) \xrightarrow{\cong} H_k((P_{n-1}X)^+, (P_n X)^+; \mathbb{Z}). \quad ∎$$

3.2 Lemma: *For $n \geq 2$, F is $(n-1)$-connected.*

Proof: F is at least $(n-2)$-connected because $K(\pi_n X, n)$ and Φ are $(n-1)$-connected. We must show that $\pi_{n-1} F = 0$ as well. First of all, we have an epimorphism $\pi_n \Phi \twoheadrightarrow \pi_{n-1} F$. So $\pi_{n-1} F$ is abelian, and the Hurewicz map $\pi_{n-1} F \xrightarrow{\cong} H_{n-1} F$ is an isomorphism even for $n = 2$. Next, by applying the

Serre spectral sequence to the fibration $AP_nX \to AP_{n-1}X$, we see that $H_{n-1}F$ is a 1-acyclic $\widetilde{G}$-module. Furthermore, the commutative diagram

$$\begin{array}{ccc} \pi_{n-1}\Omega\Phi & \twoheadrightarrow & \pi_{n-1}F \\ \downarrow & & \downarrow \\ \pi_{n-1}\Omega(P_nX)^+ & \longrightarrow & \pi_{n-1}AP_nX \end{array}$$

tells us that $\widetilde{G}$ acts trivially on the image of $\pi_{n-1}F \to \pi_{n-1}AP_nX$. But the class of 0-acyclic modules is closed under quotients by Proposition 2.12. So this image is trivial, and we have an epimorphism $\partial\colon \pi_nAP_{n-1}X \twoheadrightarrow \pi_{n-1}F$. On the other hand, $\widetilde{G}$ acts trivially on $\pi_kAP_{n-1}X$ for $k \geq n$, because we have isomorphisms $\pi_k\Omega(P_{n-1}X)^+ \xrightarrow{\cong} \pi_kAP_{n-1}X$ in the Π-central fibration $\Omega(P_{n-1}X)^+ \to AP_{n-1}X \to P_{n-1}X$. Now ∂ is a morphism of $\widetilde{G}$-modules, implying that $\widetilde{G}$ acts trivially on the 1-acyclic $\widetilde{G}$-module $\pi_{n-1}F$. Therefore $\pi_{n-1}F = 0$, as claimed. ∎

Thus we have established the first part of Theorem A. We now turn to diagram (UCE) of the Theorem and its properties:

The bottom row comes from the fibration $F \to K(\pi_nX, n) \to \Phi$, using Lemma 3.2. The terms $\pi_{n+1}\Phi$ and $\pi_n\Phi$ are trivial $\widetilde{G}$-modules and $\pi_{n+1}\Phi$ is contained in the center of π_nF. Further, $\pi_nF \cong H_nF$ is seen to be a 1-acyclic $\widetilde{G}$-module, by using the Serre spectral sequence of the fibration $F \to AP_nX \to AP_{n-1}X$. Thus $N := \mathrm{im}(\pi_nF \to \pi_nX) = I[\widetilde{G}].\pi_nX = \mathrm{im}(\mu)$ is the maximal perfect submodule of π_nX; see Theorem D(i). From Theorem 2.10 we see that $\pi_{n+1}\Phi \rightarrowtail \pi_nF \twoheadrightarrow N$ is the universal central extension of N. So the vertical arrows on the left are isomorphisms by Theorem D(ii). The vertical arrow on the right is an isomorphism by the Five Lemma.

As to $\pi_{n+1}F$, it is a trivial $\widetilde{G}$-module because it fits into the exact sequence $\pi_{n+2}AP_{n-1}X \to \pi_{n+1}F \to \pi_{n+1}AP_nX$, where $\widetilde{G}$ acts trivially on the outside terms. The Hurewicz map $\pi_{n+1}F \twoheadrightarrow H_{n+1}F$ is onto and is a $\widetilde{G}$-module map. Thus $\widetilde{G}$ acts trivially on $H_{n+1}F$ as well. Now the Serre spectral sequence yields an isomorphism

$$H_2(\widetilde{G}; I \otimes_{\widetilde{G}} \pi_nX) \xrightarrow{\cong} H_0(\widetilde{G}; H_{n+1}F) \cong H_{n+1}F,$$

which proves the claim, and completes the proof of Theorem A. ∎

3.3 Remark: By chasing the diagram of homotopy groups coming from the fibration diagram (FD) one can deduce further that the maximal perfect submodule of $\pi_n X$ is always contained in $\ker(\pi_n P_n X \to \pi_n(P_n X)^+)$. Moreover, the two modules are equal exactly when $\pi_{n+1}(P_n X)^+ \to \pi_{n+1}(P_{n-1}X)^+$ is onto. ∎

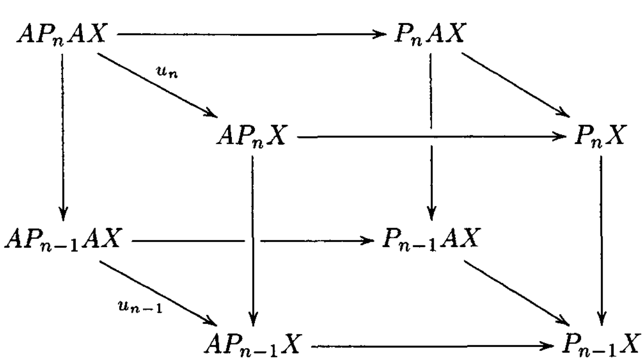
4. The acyclic Postnikov tower of AX

Already in the early 1970's Dror showed how to use the acyclic Postnikov tower [3] to analyze an acyclic space Z. The acyclic Postnikov n-stage of Z is simply the acyclization $AP_n Z$ of the usual Postnikov section. The acyclic Postnikov n-stage need not have trivial homotopy groups above dimension n. Instead, the only requirement is that the fundamental group act trivially on these higher homotopy groups.

When passing from an $(n-1)$-stage Z_{n-1} to an n-stage, one splices into $\pi_* Z_{n-1}$ a 1-acyclic $\pi_1 Z$-module α_n, and there is a corresponding "acyclic Postnikov invariant" $\kappa_n \in H^{n+1}(Z_{n-1}; M)$. In addition, in dimensions greater than n, one splices into $\pi_* Z_{n-1}$ certain $\pi_1 Z$-modules with trivial action.

In general, starting with an arbitrary space X, Dror's *acyclic Postnikov tower* of AX has $AP_n AX$ as its n-th acyclic Postnikov stage. In Theorem A, we were working with a tower whose n-th stage is $AP_n X$. Below, we establish explicitly a natural equivalence between these towers. With the aid of Theorem A, we express the acyclic Postnikov invariants of AX in terms of the ordinary Postnikov invariants of X.

4.1 LEMMA: *Let X be a connected CW-space. Applying successively the appropriate functors to the map $AX \to X$ yields the commutative cube*



whose left hand face is a homotopy equivalence of acyclic Postnikov towers.

Proof: To see that each u_n is a homotopy equivalence, we argue as follows.

Applying A to the commutative diagram

$$\begin{array}{ccc} AX & \longrightarrow & X \\ \downarrow & & \downarrow \\ P_nAX & \longrightarrow & P_nX \end{array} \quad \text{yields} \quad \begin{array}{ccc} AAX & \xrightarrow{\cong} & AX \\ \downarrow & & \downarrow \\ AP_nAX & \xrightarrow[u_n]{} & AP_nX \end{array}$$

For $k \leq n$, the right hand square induces π_k-isomorphisms because the maps on the top and the sides do. This follows from Lemma 3.2. By [3, 3.4], u_n is a homotopy equivalence. ∎

4.2 COROLLARY: *The functors AP_nA and AP_n are naturally equivalent.*

In order to determine the acyclic Postnikov invariants of AX, we require the following cohomological recognition tool for acyclic spaces:

4.3 LEMMA: *A connected CW-space X is acyclic if and only if its fundamental group G is 2-acyclic and, for every G-module M, the morphism μ: $I \otimes_G M \to M$ induces isomorphisms*

$$\mu_*: H^r(X; I \otimes_G M) \xrightarrow{\cong} H^r(X; M) \quad \text{for } r \geq 2.$$

Proof: If X is acyclic, then G is 2-acyclic; see [3, 4.1]. To see that μ_* is an isomorphism, we split the sequence (MS) up into short exact sequences

$$H_1(G; M) \rightarrowtail I \otimes_G M \twoheadrightarrow PM \quad \text{and} \quad PM \rightarrowtail M \twoheadrightarrow H_0(G; M),$$

where PM denotes the maximal perfect submodule of M; see Theorem D. We then get coefficient sequences of the form

$$\begin{array}{ccccc} H^r(X;H_1(G;M)) \to H^r(X;I\otimes_G M) & \longrightarrow & H^r(X;PM) & \longrightarrow & H^{r+1}(X;H_1(G;M)) \\ & & \| & & \\ H^{r-1}(X;H_0(G;M)) & \longrightarrow & H^r(X;PM) & \longrightarrow & H^r(X;M) \to H^r(X;H_0(G;M)) \end{array}$$

The coefficient map μ_* appears as a composite in the middle of the diagram. If X is acyclic, then the end terms of both rows are 0. So μ_* is an isomorphism.

Now suppose G is 2-acyclic and μ_* is an isomorphism for all M and $r \geq 2$. With $M = \mathbb{Z}[G]$ we have $H_1(G; M) = 0$ and, consequently, isomorphisms $H^r(X; I) \xrightarrow{\cong} H^r(X; PM)$ for all $r \geq 2$. So $H^r(X; PM) \xrightarrow{\cong} H^r(X; \mathbb{Z}[G])$ are isomorphisms for $r \geq 2$ as well. We have $H^1(X; \mathbb{Z}) = H^1(G; \mathbb{Z}) = 0$. But then $H^r(X; \mathbb{Z}) = 0$ for $r \geq 1$. So X is acyclic. ∎

4.4 Proposition: *Let X be a connected CW-space with n-th k-invariant k_n in $H^{n+1}(P_{n-1}X;\pi_nX)$. Then the n-th acyclic k-invariant of AX (see [3]) is $\mu^{-1}\circ c_{n-1}(k_n)$:*

$$H^{n+1}(P_{n-1}X;\pi_nX)\xrightarrow{c_{n-1}}H^{n+1}(AP_{n-1}X;\pi_nX)\xrightarrow{\mu^{-1}}H^{n+1}(AP_{n-1}X;I\otimes_G\pi_nX).$$

Here c_{n-1}: $AP_{n-1}X\to P_{n-1}X$ is the colocalizing map, G is π_1AX, I is the augmentation ideal of $\mathbb{Z}[G]$, and μ^{-1} is the coefficient isomorphism of Lemma 4.3.

Sketch of Proof: Consider the fibration $Y\to AP_{n-1}X$ obtained from the proposed acyclic k-invariant. There is a morphism of fibrations φ: $AP_nX\to Y$ over $AP_{n-1}X$. With the methods supplied in the previous discussion it is possible to show that

(1) $\pi_r\varphi$ is an isomorphism for $1\le r\le n$;
(2) $\pi_r(AY\to Y)$ is an isomorphism for $1\le r\le n$;
(3) the unique lift f: $AP_nX\to AY$ of φ is a weak homotopy equivalence.

This implies the claim. ∎

4.5 Remark: In many situations our work can be used to clarify the effect on homotopy groups of plus constructions and localizations with respect to more general homology theories h. For example, let h be connective. Note first that $X\to X^h$ (the h-homology localization of X) factors through $X\to X^+$. If X^+ is simply connected, then the canonical map $X^{+h}\to X^h$ is a homotopy equivalence; see [9, 1.7]. Now $X\to X^{+h}$ agrees with $X\to X^{+HR}$ for a suitable ring R of the form $\mathbb{Z}_P$ or $\bigoplus_{p\in P}\mathbb{Z}/p$, where P is a set of primes; see [1, 1.1] and compare [12, Sect. 4].

Consequently, the four localization maps

$$X\to X^h,\quad X\to X^{+h},\quad X\to X^{HR},\quad X\to X^{+HR}$$

all agree and factor as $X\xrightarrow{u}X^+\xrightarrow{v}(X^+)^{HR}$. The map π_*v is completely understood by [2], and here we provide new information on π_*u.

References

[1] A. K. Bousfield, *Types of acyclicity*, Journal of Pure and Applied Algebra **4** (1974), 293–298.

[2] A. K. Bousfield, *The localization of spaces with respect to homology*, Topology **14** (1975), 135–150.

[3] E. Dror, *Acyclic spaces*, Topology **11** (1972), 339–348.

[4] E. Dror Farjoun, *Cellular Spaces, Null Spaces, and Homotopy Localization*, Lecture Notes in Mathematics **1622**, Springer-Verlag, Berlin–New York, 1995.

[5] J. C. Hausmann and D. Husemoller, *Acyclic maps*, L'Enseignement Mathématique **25** (1979), 53–75.

[6] M. A. Kervaire, *Multiplicateurs de Schur et K-théorie*, in *Essays on Topology and Related Topics: Memoires dédiés à Georges de Rham (1903–)* (A. Haefliger and R. Narasimhan, eds.), Springer-Verlag, Berlin–New York, 1970, pp. 212–225.

[7] W. Meier and R. Strebel, *Homotopy groups of acyclic spaces*, The Quarterly Journal of Mathematics. Oxford **32** (1981), 81–95.

[8] J. W. Milnor, *Introduction to Algebraic K-Theory*, Annals of Mathematics Studies **72**, Princeton University Press, Princeton, NJ, 1971.

[9] G. Mislin and G. Peschke, *Central extensions and generalized plus constructions*, Transactions of the American Mathematical Society **353** (2001), 585–608.

[10] D. G. Quillen, *Cohomology of groups*, in *Actes du Congrès International des Mathématiciens (Nice, 1970)*, Volume II, Gauthier Villars, Paris, 1971, pp. 47–51.

[11] D. G. Quillen, *Higher K-theory for categories with exact sequences*, in *New Developments in Topology (Proceedings of Symposia in Algebraic Topology, Oxford, 1972)*, London Mathematical Society Lecture Notes **11**, Cambridge University Press, 1972, pp. 95–103.

[12] J.-Y. Tai, *Generalized plus-constructions and fundamental groups*, Journal of Pure and Applied Algebra **132** (1998), 207–220.